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THE ORIENTED ELASTIC CONTINUUM AS A MODEL FOR THE MAGNETOELASTIC BODY†

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Abstract—The elastic continuum with a single director field is used as a model for the description of magneto-elastic interactions in solids, the director being identified with the magnetic moment/unit mass. The theory is formulated for a continuum of grade 2 and the field equations and boundary conditions are derived from a variational principle. One arrives at non-symmetric stresses and couple-stresses which are of mechanical and magnetic origin.

1. INTRODUCTION

IN CONNEXION with the development of the theory of elastic dielectrics, investigations of the coupling of a magnetic field with a deformation field have been carried out during the recent years.

In 1964, Tiersten [1] derived the differential equations and boundary conditions for a non-conducting, magnetically saturated elastic medium by means of a systematic application of the laws of continuum physics to a model consisting of an electronic spin continuum coupled to a lattice continuum. These results were later deduced by Tiersten [2] from a variational principle. Brown [3] gives a summary of the methods used in the field of magnetoelastic interactions. Rieder [4] pointed out that the Cosserat continuum may serve as an appropriate model for the description of magnetoelastic effects. This model has been applied by Alblas [5, 6] to a series of problems related to the deformation of magnetic materials.

In this paper the magnetoelastic solid is treated as an elastic continuum with a director field, the director being identified with the magnetic moment/unit mass. The field equations and boundary conditions are derived from a variational principle. We shall restrict ourselves to the case of magnetostatics and shall neglect thermodynamic effects. Besides the energy of translation of the material points we shall introduce into the kinetic energy the energy of the director field. We shall use the expression given in the theory of polar media for the case of a single director whereas Alblas [5] started from a term which is known from the theory of micromagnetics. It will be shown that for a special case the generalized moment of inertia occurring in the director energy, may be associated with well-known physical quantities. By means of the director energy we succeed in defining a spin density for the material. Additionally the theory is formulated for a medium of grade 2, i.e. the specific internal energy is a function of the second deformation gradient, besides other variables. In such a material couple-stresses are found which are of mechanical as well as of magnetic origin; the stress tensor becomes non-symmetric. It will be shown that the requirement of

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invariance of the specific internal energy under rigid rotations of the spatial observer system leads to the dynamic equations of the Cosserat continuum.

2. THE MAGNETIC AND KINEMATICAL STATE OF THE MEDIUM

Confining ourselves to the case of magnetostatics and excluding the presence of electric fields, charges and currents, Maxwell's equation for the magnetic (Maxwell) field $\boldsymbol{H}$ valid in the material volume v_M as well as in free space v_F, is given by

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{0}. \tag{1}$$

Hence the Maxwell field $\boldsymbol{H}$ may be expressed as the gradient of the magnetic scalar potential φ

$$\boldsymbol{H} = -\operatorname{grad} \varphi. \tag{2}$$

The boundary condition for the magnetic field $\boldsymbol{H}$ on the surface s of the material body under the given restrictions is

$$\boldsymbol{n} \times [\![\boldsymbol{H}]\!] = \boldsymbol{0}, \tag{3}$$

where $[\![\boldsymbol{H}]\!]$ denotes, as usual, the jump across s of $\boldsymbol{H}$ from exterior to interior and $\boldsymbol{n}$ is the exterior normal unit vector of the surface s.

The magnetic state of the material is described by means of the magnetization $\boldsymbol{M}$, the magnetic moment/unit volume. The macroscopic fields are linked to each other by the magnetic constitutive equation (in Gaussian units)

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}, \tag{4}$$

where $\boldsymbol{B}$ is the magnetic induction. As the magnetization vanishes in free space, we have

$$\boldsymbol{B} = \boldsymbol{H} \tag{5}$$

in v_F. In the case of a deformable magnetic body it is appropriate to introduce the magnetic moment $\boldsymbol{\mu}$/unit mass instead of the magnetization:

$$\boldsymbol{\mu} := \frac{1}{\rho} \boldsymbol{M}; \tag{6}$$

ρ is the mass density of the material.

In a magnetically saturated body, the magnitude of the magnetic moment/unit mass is conserved since the mass is conserved; the magnetic moment merely undergoes a rigid rotation. In the saturation condition

$$\boldsymbol{\mu} \cdot \boldsymbol{\mu} = \mu_s^2 = \text{const.}, \qquad \boldsymbol{\mu} \cdot \delta \boldsymbol{\mu} = 0 \tag{7}$$

μ_s is the saturation value of the magnetic moment. For instance a paramagnetic body is magnetically saturated. In the case of a ferromagnetic solid, we may identify the Weissian domains in which the molecular magnetic moments are lined up parallely, with the mass elements of the medium and regard $\boldsymbol{\mu}$ as the magnetic moment of the domains. Then the rotation of the magnetic moments into the direction of the magnetic field can be described by the presented theory, but not the motion of the domain boundaries (Bloch walls) which in general takes place already at lower field magnitudes.

We shall make use of the oriented elastic continuum as a model for the magnetoelastic medium by attaching to each material point a single director which will be identified with the magnetic moment $\boldsymbol{\mu}$/unit mass.

In the reference configuration at time $t = t_0$ the position of a material point P is represented by means of the material coordinates X^K and the magnetic moment associated with the material point P will be denoted by $\mathring{\boldsymbol{\mu}}$. In the deformed instantaneous configuration at $t > t_0$ the position of the same material point P is fixed by the spatial coordinates x^k, and the magnetic moment which in general will have rotated with respect to the reference configuration, is denoted by $\boldsymbol{\mu}$ (Fig. 1).

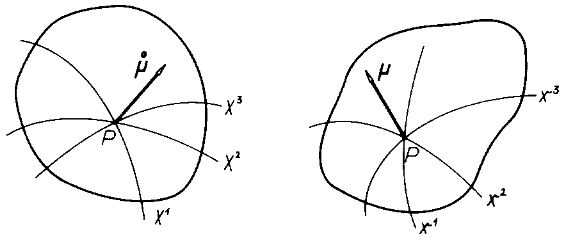

FIG. 1.

When the quantities are referred to the reference configuration, their indices will be majuscules, and when they are referred to the instantaneous configuration, their indices will be minuscules [7].

The motion of the body is given by the one-parameter families of mappings

$$x^k = x^k(X^K, t), \tag{8}$$

$$\mu^k = \mu^k(\mathring{\mu}^K(X^L), X^K, t) = \mu^k(X^K, t), \tag{9}$$

where the rotation of the director is independent of the displacement of the material point to which it is attached. In order that the axiom of continuity be fulfilled it is necessary that the Jacobian j of (8) does not vanish:

$$0 < j = \det[x^k_{;K}] < \infty. \tag{10}$$

The deformation of the point continuum is described by means of the (first) deformation gradient $x^k_{;K}$ and the second deformation gradient $x^k_{;KL}$, the semicolon denoting the total covariant derivative [7]. The distortion of the director field is distinguished by the magnetization gradient $\mu^k_{;K}$ (director gradient). Besides other variables, these three deformation measures will be used later as state variables in the specific internal energy. The inclusion of the magnetization gradient allows to describe the exchange energy in ferromagnetic materials or to take into account a magnetic interaction which may become significant in a body with a strong inhomogeneous distribution of magnetic moments.

3. THE VARIATIONAL PRINCIPLE

To derive the differential equations and boundary conditions, we start with Hamilton's principle

$$\delta \int_{t_1}^{t_2} (T-U)\,\mathrm{d}t + \int_{t_1}^{t_2} \delta A\,\mathrm{d}t = 0, \tag{11}$$

where T is the kinetic energy of the material body, U the potential energy and δA the virtual work of the impressed forces. In the instantaneous configuration, the field quantities $\boldsymbol{x}$, $\boldsymbol{\mu}$ and φ are varied:

$$\begin{aligned} &\boldsymbol{x} \to \boldsymbol{x}+\delta\boldsymbol{x}, \\ &\boldsymbol{\mu} \to \boldsymbol{\mu}+\delta\boldsymbol{\mu} \quad \text{with } \boldsymbol{\mu}\,.\,\delta\boldsymbol{\mu} = 0. \\ &\varphi \to \varphi+\delta\varphi. \end{aligned} \tag{12}$$

The kinetic energy consists of the translational energy of the material points and the kinetic energy of the director field:

$$T = \int_{v_M} \tfrac{1}{2}\rho\dot{\boldsymbol{x}}\,.\,\dot{\boldsymbol{x}}\,\mathrm{d}v + \int_{v_M} \tfrac{1}{2}\rho I\dot{\boldsymbol{\mu}}\,.\,\dot{\boldsymbol{\mu}}\,\mathrm{d}v. \tag{13}$$

The director energy follows from the expression in the general case of an arbitrary number of directors [8] by considering that only a single director occurs here, the magnetic moment/unit mass. Since we are dealing with a rigid director, the director energy may be interpreted as the kinetic energy of rotation of the magnetic moments. As will be shown later, we may define a spin—or angular momentum density with the aid of the director energy. For the sake of simplification we shall assume that the generalized moment of inertia I is constant with respect to space and time. We shall see later that for a special case we succeed in associating known and measurable physical quantities with the generalized moment of inertia.

The potential energy is composed of a local or short-range part which is to contain, besides the elastic energy, the interaction energy of the magnetic moments with the lattice (magnetoelastic energy) and the interaction energy of adjacent magnetic moments and a long-range part comprising the magnetic field energy in the entire space v and the interaction energy of the magnetic moments with the macroscopic Maxwell field in the material volume v_M. If we describe the local part of the potential energy by means of the specific internal energy ε, we obtain in the case of permanent magnetic moments:

$$U = \int_{v_M} \rho\varepsilon\,\mathrm{d}v - \frac{1}{2}\frac{1}{4\pi}\int_{v} \boldsymbol{H}\,.\,\boldsymbol{H}\,\mathrm{d}v - \int_{v_M} \boldsymbol{H}\,.\,\boldsymbol{M}\,\mathrm{d}v. \tag{14}$$

In the general electrodynamic case the long-range electromagnetic interaction cannot be taken into account in this simple manner. It must then be introduced into the variational principle by the virtual work of electromagnetic forces.

The specific internal energy be the following state function:

$$\varepsilon = \hat{\varepsilon}(x^k_{;K}, x^k_{;KL}, \mu^k, \mu^k_{;K}, X^K, \boldsymbol{G}_K), \tag{15}$$

where the dependence of ε on the material coordinates X^K denotes a possible inhomogeneity of the material and the dependence on the base vectors $\boldsymbol{G}_K$ of the reference configuration describes symbolically a possible anisotropy of the medium. According to (15) we are dealing with an oriented hyperelastic material of grade 2.

To simplify the theory, we shall insert into the virtual work δA of the impressed forces only the work associated with the mechanical body force $\boldsymbol{f}$ and the mechanical surface traction $\boldsymbol{t}$, and shall not take into account the work associated with mechanical body— and surface couples:

$$\delta A = \int_{v_M} \boldsymbol{f}\,.\,\delta\boldsymbol{x}\,\mathrm{d}v + \oint_s \boldsymbol{t}\,.\,\delta\boldsymbol{x}\,\mathrm{d}s. \tag{16}$$

If we introduce into the variational principle the condition (7) of magnetic saturation in the material volume v_M and on the surface s of the body by means of Lagrangian multipliers λ and κ, Hamilton's principle finally may be written in the form:

$$\delta\int_{t_1}^{t_2}\mathrm{d}t\int_{v_M}\{\tfrac{1}{2}\rho\dot{\boldsymbol{x}}\,.\,\dot{\boldsymbol{x}}+\tfrac{1}{2}\rho I\dot{\boldsymbol{\mu}}\,.\,\dot{\boldsymbol{\mu}}\}\,\mathrm{d}v-\delta\int_{t_1}^{t_2}\mathrm{d}t\left\{\int_{v_M}\rho\varepsilon\,\mathrm{d}v-\frac{1}{2}\frac{1}{4\pi}\int_v \boldsymbol{H}\,.\,\boldsymbol{H}\,\mathrm{d}v-\int_{v_M}\boldsymbol{H}\,.\,\boldsymbol{M}\,\mathrm{d}v\right\}$$
$$+\int_{t_1}^{t_2}\mathrm{d}t\left\{\int_{v_M}\boldsymbol{f}\,.\,\delta\boldsymbol{x}\,\mathrm{d}v+\oint_s\boldsymbol{t}\,.\,\delta\boldsymbol{x}\,\mathrm{d}s\right\}+\int_{t_1}^{t_2}\mathrm{d}t\left\{\int_{v_M}\lambda\boldsymbol{\mu}\,.\,\delta\boldsymbol{\mu}\,\mathrm{d}v+\oint_s\kappa\boldsymbol{\mu}\,.\,\delta\boldsymbol{\mu}\,\mathrm{d}s\right\}=0. \tag{17}$$

4. DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

When considering at the performance of variation that [2]

$$\delta H_i = -\delta(\varphi_{,i}) = -(\delta\varphi)_{,i}+\varphi_{,k}(\delta x^k)_{,i}, \tag{18}$$

after some manipulations Hamilton's principle assumes the following form:

$$\int_{t_1}^{t_2}\mathrm{d}t\left\{\int_{v_M}\left[-\rho\ddot{x}_i+\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;K}}x^k_{;K}+\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;KL}\right)_{,k}-\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;K}x^l_{;L}\right)_{,lk}\right.\right.$$
$$\left.+\frac{1}{4\pi}\left(H_iB^k-\frac{1}{2}H_lH^l\delta^k_i\right)_{,k}+f_i\right]\delta x^i\,\mathrm{d}v+\int_{v_M}\left[-\rho I\ddot{\mu}_i+\left(\rho\frac{\partial\varepsilon}{\partial\mu^i_{;K}}x^k_{;K}\right)_{,k}\right.$$
$$\left.-\rho\frac{\partial\varepsilon}{\partial\mu^i}+\rho H_i+\lambda\mu_i\right]\delta\mu^i\,\mathrm{d}v+\int_{v_M}\frac{1}{4\pi}B^l_{,l}\,\delta\varphi\,\mathrm{d}v+\int_{v_F}\frac{1}{4\pi}B^l_{,l}\,\delta\varphi\,\mathrm{d}v$$
$$+\int_{v_F}\frac{1}{4\pi}\left(H_iB^k-\frac{1}{2}H_lH^l\,\delta^k_i\right)_{,k}\delta x^i\,\mathrm{d}v+\oint_s\left[n_k\left\{\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;K}x^l_{;L}\right)_{,l}\right.\right.$$
$$\left.-\rho\frac{\partial\varepsilon}{\partial x^i_{;K}}x^k_{;K}-\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;KL}+\frac{1}{4\pi}\left[\!\!\left[H_iB^k-\frac{1}{2}H_lH^l\,\delta^k_i\right]\!\!\right]\right\}+D_k\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;K}x^l_{;L}n_l\right)$$
$$\left.+b^m{}_m\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;K}x^l_{;L}\right)n_kn_l+t_i\right]\delta x^i\,\mathrm{d}s-\oint_s\left[\left(\rho\frac{\partial\varepsilon}{\partial\mu^i_{;K}}x^k_{;K}\right)n_k-\kappa\mu_i\right]\delta\mu^i\,\mathrm{d}s$$
$$\left.+\oint_s\frac{1}{4\pi}n_l[\![B^l]\!]\,\delta\varphi\,\mathrm{d}s-\oint_s\left(\rho\frac{\partial\varepsilon}{\partial x^i_{;KL}}x^k_{;K}x^l_{;L}\right)n_kn_lD(\delta x^i)\,\mathrm{d}s\right\}=0. \tag{19}$$

Here $\boldsymbol{b}$ is the second fundamental form of the surface s which is supposed to have no edges, D_k is the surface gradient and $D(\delta x^i)$ is the normal derivative of δx^i on s [9, 10].

Before stating the field equations and boundary conditions, we shall first define the following field tensors:

The local stress tensor

$$ {}_L t_i{}^k := \rho \frac{\partial \varepsilon}{\partial x^i_{;K}} x^k_{;K} + \rho \frac{\partial \varepsilon}{\partial x^i_{;KL}} x^k_{;KL} - \left(\rho \frac{\partial \varepsilon}{\partial x^i_{;KL}} x^k_{;K} x^l_{;L} \right)_{,l} . \tag{20} $$

The magnetic (Maxwell) stress tensor

$$ {}_M \boldsymbol{t} := \frac{1}{4\pi} (\boldsymbol{H} \otimes \boldsymbol{B} - \tfrac{1}{2} \boldsymbol{H} . \boldsymbol{H} \mathbf{1}). \tag{21} $$

The generalized stress tensor

$$ \boldsymbol{t} := {}_L \boldsymbol{t} + {}_M \boldsymbol{t}. \tag{22} $$

The hyperstress tensor

$$ h_i{}^{kl} := \rho \frac{\partial \varepsilon}{\partial x^i_{;KL}} x^k_{;K} x^l_{;L} . \tag{23} $$

The local magnetic field

$$ {}_L H_i := \frac{1}{\rho} \left[\left(\rho \frac{\partial \varepsilon}{\partial \mu^i_{;K}} x^k_{;K} \right)_{,k} - \rho \frac{\partial \varepsilon}{\partial \mu^i} \right] . \tag{24} $$

The local magnetic field tensor

$$ {}_L H_i{}^k := \rho \frac{\partial \varepsilon}{\partial \mu^i_{;K}} x^k_{;K} . \tag{25} $$

The Lagrangian multipliers λ and κ in the differential equation and boundary condition resulting from the variation $\delta \boldsymbol{\mu} \neq \mathbf{0}$ are eliminated by forming the dyad of these equations with $\boldsymbol{\mu}$ and retaining only the antisymmetric parts. When we further introduce the magnetic spin—or angular momentum density

$$ S_{ki} := I \mu_{[i} \dot{\mu}_{k]} \tag{26a} $$

or in the dual vector notation

$$ \boldsymbol{S} := I \boldsymbol{\mu} \times \dot{\boldsymbol{\mu}}, \tag{26b} $$

we finally obtain the following differential equations and boundary conditions:

$$ \rho \ddot{\boldsymbol{x}} = \operatorname{div} \boldsymbol{t} + \boldsymbol{f}, \tag{27} $$

$$ \dot{\boldsymbol{S}} = \boldsymbol{\mu} \times (\boldsymbol{H} + {}_L \boldsymbol{H}), \qquad \text{in } v_M \tag{28} $$

$$ \operatorname{div} \boldsymbol{B} = 0, \tag{29} $$

$$ \operatorname{div} {}_M \boldsymbol{t} = \mathbf{0}, \qquad \text{in } v_F \tag{30} $$

$$ \operatorname{div} \boldsymbol{B} = 0, \tag{31} $$

$$ [\![t_i{}^k]\!] n_n + D_k (h_i{}^{kl} n_l) = 0, \tag{32} $$

$$ h_i{}^{kl} n_n n_l = 0, \tag{33} $$

$$ \mu_{[i\,L} H_{k]}{}^l n_l = 0, \qquad \text{on } s \tag{34} $$

$$ \boldsymbol{n} . [\![\boldsymbol{B}]\!] = 0. \tag{35} $$

In the first boundary condition (32) the second condition (33) has already been used and the jump of the generalized stress tensor $\boldsymbol{t}$ across s has been defined as

$$[\![t_i^{\ k}]\!]n_k := (t_i - {}_Lt_i^{\ k}n_k) + [\![{}_Mt_i^{\ k}]\!]n_k. \tag{36}$$

To this system of field equations and boundary conditions, equations (1) and (3)–(5) have to be appended.

Equation (27) is the equation of motion for the magnetoelastic body. The generalized stress tensor $\boldsymbol{t}$ is defined as the sum of the local stress tensor ${}_L\boldsymbol{t}$ and the magnetic stress tensor ${}_M\boldsymbol{t}$, from the divergence of which the magnetic body force

$${}_M\boldsymbol{f} := \operatorname{div} {}_M\boldsymbol{t} = (\boldsymbol{M}\,.\,\operatorname{grad})\boldsymbol{H} \tag{37}$$

follows. The antisymmetric part $(-{}_M\boldsymbol{t}^A)$ of the magnetic stress tensor to which the dual vector ${}_M\boldsymbol{l}$ may be appointed, yields the magnetic body couple

$${}_M\boldsymbol{l} = \boldsymbol{M} \times \boldsymbol{H}. \tag{38}$$

The field equation (28) represents the equation of magnetic angular momentum (balance of moment of momentum for the spin continuum): the time rate of the spin density equals the torque on the magnetic moment $\boldsymbol{\mu}$ in the field $(\boldsymbol{H} + {}_L\boldsymbol{H})$.

Equation (29) or (31) is the Maxwell equation for the magnetic induction $\boldsymbol{B}$.

Since the magnetization vanishes in free space, equation (30) is identically fulfilled according to (37).

Equations (32)–(35) represent boundary conditions on the surface of the material body for the generalized stress tensor, the hyperstress tensor, the local magnetic field tensor and the magnetic induction.

5. THE EQUATION OF MOMENT OF MOMENTUM

The field equation (28) can be expressed in another form which is more familiar to continuum mechanics. If we require that the specific internal energy is invariant under rigid rotations of the spatial observer system, from a theorem of Weyl [11] follows that ε must satisfy the condition

$$\frac{\partial \varepsilon}{\partial x_{[i;K}} x^{k]}_{;K} + \frac{\partial \varepsilon}{\partial x_{[i;KL}} x^{k]}_{;KL} + \frac{\partial \varepsilon}{\partial \mu_{[i}} \mu^{k]} + \frac{\partial \varepsilon}{\partial \mu_{[i;K}} \mu^{k]}_{;K} = 0. \tag{39}$$

It is then easy to verify that

$${}_Lt^{[ik]} + {}_Mt^{[ik]} + (h^{[ik]l} + {}_LH^{[i|l}\mu^{k]})_{,l} = \rho(H^{[i} + {}_LH^{[i})\mu^{k]} \tag{40}$$

holds. We now define the couple–stress tensor

$$m^{ikl} := h^{[ik]l} + {}_LH^{[i|l}\mu^{k]}, \tag{41}$$

where to the first term known from the materials of grade 2, an additional part has been added which is brought about by the dependence of the internal energy on the magnetization gradient. Replacing the right side of (40) by the time rate of the spin density according to (28), we finally obtain

$$\rho \dot{S}^{ik} = t^{[ik]} + m^{ikl}_{\ \ ,l}. \tag{42}$$

In the case of vanishing spin density (i.e. vanishing director energy) this form of the equation of moment of momentum reduces to the Cosserat equation

$$t^{[ik]}+m^{ikl}{}_{,l} = 0, \tag{43}$$

which links the antisymmetric part of the stress tensor to the divergence of the couple–stress tensor. Consequently, in the case of a symmetric generalized stress tensor (vanishing spin density and vanishing divergence of the couple–stress tensor) the antisymmetric part of the local stress tensor must equal the negative of the antisymmetric part of the Maxwell stress tensor (38).

From micromagnetics the equation of angular momentum is known in the form [12]

$$-\frac{1}{\gamma}\dot{\boldsymbol{\mu}} = \boldsymbol{\mu}\times \boldsymbol{H}_{\text{eff}}, \tag{44}$$

where γ is the gyromagnetic ratio and $\boldsymbol{H}_{\text{eff}}$ the so-called effective magnetic field intensity.

Let us now consider the simple special case

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\omega}\times\boldsymbol{\mu}, \tag{45}$$

where $\boldsymbol{\omega}$ is the constant vector of angular velocity of the Larmor precession. We then have

$$\boldsymbol{\mu}\times\ddot{\boldsymbol{\mu}} = -(\boldsymbol{\omega}\,.\,\boldsymbol{\mu})\dot{\boldsymbol{\mu}}. \tag{46}$$

Substituting this result into the equation (28) of angular momentum and identifying $\boldsymbol{H}_{\text{eff}}$ with the field $(\boldsymbol{H}+{}_L\boldsymbol{H})$ yields

$$\dot{\boldsymbol{S}} = I\boldsymbol{\mu}\times\ddot{\boldsymbol{\mu}} = -I(\boldsymbol{\omega}\,.\,\boldsymbol{\mu})\dot{\boldsymbol{\mu}} = \boldsymbol{\mu}\times\boldsymbol{H}_{\text{eff}}, \tag{47}$$

and by comparison of coefficient with equation (44) we find

$$I = \frac{1}{\gamma}\frac{1}{\boldsymbol{\omega}\,.\,\boldsymbol{\mu}}. \tag{48}$$

Thus in the special case of the Larmor precession the generalized moment of inertia can be determined from the gyromagnetic ratio, the Larmor frequency and the magnetic moment/unit mass.

Alblas [5] has shown how to arrive at the Einstein–de Haas effect by formulating the balance of moment of momentum for the whole body.

6. THE SPECIFIC INTERNAL ENERGY AND CONSTITUTIVE EQUATIONS

The invariance of the specific internal energy under the group of Euclidean displacements of the spatial observer frame [principle of objectivity; the invariance of ε under a rigid rotation has already been used to derive (39)] implies that ε cannot depend on the spatial coordinates x^k and on time t, as has already been assumed in (15) and that the specific internal energy can be expressed as [1, 9]

$$\varepsilon = \hat{\varepsilon}(N_K, C_{KL}, D_{KL}, Q_{KLM}\,; H_\alpha^{KLM\cdots}), \tag{49}$$

where the following 36 variables have been defined:

$$N_K := g_{kl} x^k_{;K} \mu^l, \tag{50}$$

$$C_{KL} := g_{kl} x^k_{;K} x^l_{;L}, \tag{51}$$

$$D_{KL} := g_{kl} \mu^k_{;K} x^l_{;L}, \tag{52}$$

$$Q_{KLM} := g_{kl} x^k_{;K} x^l_{;LM}. \tag{53}$$

Here the anisotropy of the material is described with the aid of the material tensors $H^{KLM\cdots}_\alpha$ (α is an enumerative index) introduced by Toupin [13] (in place of the base vectors $\boldsymbol{G}_K$ of the reference configuration). They are invariant tensors of the group characterizing the point symmetry of the material in the reference configuration. In a homogeneous material these tensors are spatially constant.

Instead of Green's deformation tensor C_{KL} the Lagrangian deformation tensor E_{KL} may be employed:

$$E_{KL} := \tfrac{1}{2}(C_{KL} - G_{KL}). \tag{54}$$

Tiersten [1] has shown that the number of variables can be further reduced if the terms in the specific internal energy containing the magnetization gradient, are identified with the exchange energy [3] which produces the parallel alignment of the magnetic moments within the domains of a ferromagnetic solid. The invariance of the exchange energy under a rigid rotation of the spin system with respect to the lattice (known from the quantum mechanical description) leads to a reduction of the number of variables by 3, as in the place of the variable D_{KL} the symmetric tensor variable

$$M_{KL} := D_{KP} \overset{-1}{C}{}^{PQ} D_{LQ} = g_{kl} \mu^k_{;K} \mu^l_{;L} \tag{55}$$

can be introduced into the specific internal energy:

$$\varepsilon = \hat{\varepsilon}(N_K, E_{KL}, M_{KL}, Q_{KLM}; H^{KLM\cdots}_\alpha). \tag{56}$$

In a ferromagnetic material of grade 1 with which we shall deal in closing, the specific internal energy depends on the 15 variables N_K, E_{KL} and M_{KL} [14]:

$$\varepsilon = \hat{\varepsilon}(N_K, E_{KL}, M_{KL}; H^{KLM\cdots}_\alpha). \tag{57}$$

The specific internal energy given, the constitutive equations for the local stress tensor and the local magnetic field can be calculated:

$${}_L t^{ik} = \rho \left\{ \frac{\partial \varepsilon}{\partial E_{KL}} x^k_{;K} x^i_{;L} + \frac{\partial \varepsilon}{\partial N_K} x^k_{;K} \mu^i \right\}, \tag{58}$$

$${}_L H^i = \frac{2}{\rho} \left\{ \rho \frac{\partial \varepsilon}{\partial M_{KL}} x^k_{;K} \mu^i_{;L} \right\}_{,k} - \frac{\partial \varepsilon}{\partial N_K} x^i_{;K}. \tag{59}$$

For the specific internal energy of a ferromagnetic material we may for instance take the following polynomial approximation:

$$\rho_0\varepsilon = H_1^{KL}N_KN_L + H_2^{KL}M_{KL} + H_3^{KLM}E_{KL}N_M + H_4^{KLMN}E_{KL}N_MN_N + H_5^{KLMN}E_{KL}E_{MN} + H_6^{KLMN}E_{KL}M_{MN}, \qquad (60)$$

where the material tensors have been used as coefficients of the polynomial and have the following physical meaning:

$\boldsymbol{H}_1$, (magnetic) anisotropy tensor;
$\boldsymbol{H}_2$, exchange tensor;
$\boldsymbol{H}_3$, piezomagnetic tensor;
$\boldsymbol{H}_4$, magnetostriction tensor;
$\boldsymbol{H}_5$, elasticity tensor;
$\boldsymbol{H}_6$, exchangestriction tensor.

As $\boldsymbol{H}_3 = \boldsymbol{0}$ for an isotropic material, the theory yields the known fact that the piezomagnetic effect is found only in anisotropic media.

Brown [3] states the following polynomial approximation for the local (free) energy/unit mass of a ferromagnetic body:

$$F = F_{ex} + \tilde{F}(N_K, E_{KL}), \qquad (61)$$

where the exchange energy/unit mass is given by

$$\rho_0 F_{ex} = \frac{1}{2}\frac{1}{\mu_s^2}(H_2^{KL}M_{KL} + H_6^{KLMN}E_{KL}M_{MN}), \qquad (62)$$

when using our nomenclature, which corresponds with the expression (60) given above.

By a suitable choice of a polynomial for the specific internal energy the known constitutive equations for the magnetoelastic body and extensions or generalizations of these constitutive equations may be obtained.†

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† *Note added in proof*: the author took notice of an important paper by Tiersten [15] in which related problems are dealt with.

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Абстракт—Используется упругая сплошная среда, с одинарным полем пассивного диполя, в смысле модели для опцсания магнитноупругого взаимодействия в твердых телах. Пассивный диполь утождесмвляется с магнитным моментом на единицу мыссы. Предлагается теория для сплошной среды порядка 2. Определяются уравнения полей и граничные условия, исход из вариационного принципа. Зто прцводит к несимцетрическим пциряжсемцзц и моментным напряжениям, которых происхождение зависат от механических и магнитных свойств среды.